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On dissipation mechanisms in micromagnetics

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Abstract. Within the framework of a dynamic version of micromagnetics [1,2], the space-time evolution of magnetization $\mathbf{m}$ in a rigid, saturated ferromagnet is governed by the following equation: $\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m}\times(\mathbf{b}^{\mathrm{ni}}+\mathbf{k}+\operatorname{div}\mathbf{C})$, where the interaction couple $\mathbf{m}\times\mathbf{k}$ and the couple stress $\mathbf{C}$ are to be constitutively specified. Under constitutive assumptions for $\mathbf{k}$, $\mathbf{C}$, and the free energy ψ, that allow for equilibrium response and viscosity out of equilibrium and agree with the dissipation principle $-\mathbf{k}\cdot\dot{\mathbf{m}}+\mathbf{C}\cdot\nabla\dot{\mathbf{m}}-\dot{\psi}\geq 0$, the above evolution equation yields a broad generalization of the standard Gilbert equation. In particular, while the standard Gilbert equation only incorporates relativistic dissipation, it is shown that the dissipation mechanisms compatible with the generalized Gilbert equation include exchange dissipation [2], dry-friction dissipation [3], and others. It is also shown that the additional term proposed in [4] to account for exchange dissipation, rather than having a genuine dissipative nature, modifies instead the nature of possible equilibria; and that such a modification is an automatic side effect when dry-friction dissipation is incorporated in the manner of [3].

PACS. 75.40.Gb Dynamic properties (dynamic susceptibility, spin waves, spin diffusion, dynamic scaling, etc.) – 76.50.+g Ferromagnetic, antiferromagnetic, and ferrimagnetic resonances; spin-wave resonance

1 Introduction

A recent paper [4] lists those predictions of the "phenomenological" Landau-Lifshitz equation that are found inconsistent with both the predictions of the "microscopic" quantum-mechanical theory and the relevant experimental findings[1]. The paper contends that better predictions can be obtained if the expression for the relaxation term proposed by Landau and Lifshitz is replaced by another expression that relates relaxation to anisotropy energy and, especially, to exchange energy.

The attitude of [4] – that "phenomenological" equations should be tailored by way of analogy and *ad hoc* reasoning – is not different from Landau and Lifshitz' in their path-breaking paper [5][2]. We find nothing wrong with analogical, *ad hoc* reasoning when guided and tempered by a faultless physical insight, as was the case with Landau and Lifshitz. We find nothing especially good with it either: derivations from first principles are preferable, not only to avoid misconceptions that an analogical procedure might deem plausible but also to interpret correctly the proposed variants to a successful model.

We here use the framework of a continuum-mechanical theory of dynamic micromagnetics [1,2,7,8] to show what general, thermodynamically compatible representation the relaxation term may have in an equation that generalizes the Gilbert form [9] of the Landau-Lifshitz equation (Sect. 2). While the proposal in [4] is not compliant to such a representation, we borrow from [2] and develop a simple example of a dissipation mechanism, additional to Gilbert's, that can be regarded, in a sense that we make precise, as *exchange dissipation* (Sect. 4.1).

The general developments of Section 2 are supplemented by an analysis of the Liapounov structures associated with evolution equations of the Gilbert and Landau-Lifshitz types (Sect. 3). On the basis of this discussion we indicate that the addition to the classical Landau-Lifshitz equation of a higher-order, regularizing term of the type proposed in [4] leads to a modification of the set of possible equilibrium solutions, but leaves Gilbert's *relativistic dissipation* as the one dissipation mechanism (Sect. 4.2); and that additional dissipation is not the only outcome of including a *dry-friction* term as exemplified in [3], because the equilibrium set is modified as well (Sect. 4.3).

2 The Gilbert equation, standard and generalized

The phenomenology of the space-time evolution of a deformable ferromagnet can be described by a dynamic version of *micromagnetics* [5,10,11] that was derived from first principles in [1] and later re-exposed in [2],

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[1] The adjectives in quotes are those used in [4].

[2] Neither does the attitude differ in the exposition of Gilbert and Landau-Lifshitz equations given in Section 6.2.3 of [6], a book of many merits, where precession is introduced by analogy and dissipation by a gradient-flow argument; or in [3], p. 367, where the dry-friction dissipation mechanism discussed in Section 4.3 below is introduced by mere analogy as "a simple correction of the Landau-Lifshitz equation".

in a condensed manner, so as to cover more general constitutive circumstances.

Briefly, dynamic micromagnetics pictures a ferromagnet as the composition (not the mixture) of two interacting continua, the one with a mechanical structure, the other with a magnetic structure. The *kinematics* of the composite continuum is described by its motion with respect to a reference configuration and by the magnetization, a unit vector field over the current configuration; when the mechanical constituent continuum is modelled as rigid, as we here do, or kept immobile, the distinction between reference and current configurations need not be tracked, and a source of major difficulties is drained. The *dynamics* consists in postulating a form for the power expended in a typical process, a form where systems of generalized forces appear, which are work-conjugate to the kinematical variables. These forces are split into forces peculiar to each constituent continuum and forces that define the interaction between the two. Peculiar forces can be interior, and then they are distinguished into mutual forces, those that body parts exert on each other, and self forces, those that a body part exerts on itself; and they can be exterior, distinguished into inertial and noninertial[3]. Interaction forces act exclusively at a distance, and are supposed to have well-defined densities per unit mass. *Balance laws* are posited both for the composite continuum and the constituents, with interaction forces appearing only in the balances for the latter. The independent balances are therefore four, two of them resulting from the postulated translational invariance of the expended power, the other two from rotational invariance.

Among the balances on which the theory in [1,2] is based, the following *evolution equation for magnetic torques* has central importance:

$$\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m} \times (\mathbf{b}^{\mathrm{ni}} + \mathbf{k} + \operatorname{div}\mathbf{C}), \tag{2.1}$$

where $\mathbf{m}$ is the *magnetization* per unit mass (a unit vector), $-\gamma^{-1}\dot{\mathbf{m}}$ the *inertial couple* (with γ the gyromagnetic ratio, a nonnull material constant), $\mathbf{m} \times \mathbf{b}^{\mathrm{ni}}$ the *noninertial distance couple*, $\mathbf{m} \times \mathbf{k}$ the *interaction couple*, and $\mathbf{C}$ the *couple stress*[4].

[3] The representation issue for the inertial forces and the self-forces peculiar of the magnetic constituent continuum is discussed in [7] and [8] with greater detail than in [1] and [2].
Magnetic inertial forces lend to the balance equation of torques (Eq. (2.1) below) the *pseudoparabolic* character carried by the term on the left side: they are in fact powerless, and the truly parabolic character of (2.1) is the result of choosing a linearly viscous response out of equilibrium (*cf.* Eq. (2.14)).
The representation of magnetic self-forces is delicate because their *dipolar* nature implies that they manifest themselves as partwise-equilibrated systems of distance forces *and* contact forces: the more familiar monopolar self-forces, (typically, self-gravitation) are forces at a distance.

[4] Here and henceforth a superscript dot denotes time differentiation. Note that (2.1) guarantees the saturation condition $|\mathbf{m}| = 1$ at all times, provided it holds at the initial time. Note also that the insuppressible presence of the inertial couple makes inherently *rate-dependent* the predictions based on the evolution equation (2.1).

Later in this section we shall show how, in the Gilbert case, the term $\mathbf{m} \times \mathbf{k}$ takes the form of the Gilbert relaxation vector, and the term $\operatorname{div}\mathbf{C}$ the form of the exchange field $\alpha\Delta\mathbf{m}$. To demonstrate how equation (2.1) yields a broad generalization of the classical Gilbert equation [6,9], it is enough to restrict attention to *rigid* ferromagnets whose response is constitutively described by the mappings

$$\mathbf{k} = \hat{\mathbf{k}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \dot{\nabla}\mathbf{m}), \tag{2.2a}$$

$$\mathbf{C} = \hat{\mathbf{C}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \dot{\nabla}\mathbf{m}), \tag{2.2b}$$

for the interaction couple and the couple stress, and

$$\psi = \hat{\psi}(\mathbf{m}, \nabla\mathbf{m}) \tag{2.3}$$

for the *free energy* ψ per unit volume. This response accomodates viscosity out of equilibrium in a manner consistent with the *dissipation principle*

$$-\mathbf{k}\cdot\dot{\mathbf{m}} + \mathbf{C}\cdot\dot{\nabla}\mathbf{m} - \dot{\psi} \geq 0\,^5. \tag{2.4}$$

It can be shown that (2.2, 2.3), and (2.4) imply that the equilibrium response has the form

$$\mathbf{k}^{\mathrm{eq}} = \hat{\mathbf{k}}(\mathbf{m}, \nabla\mathbf{m}; \mathbf{0}, \mathbf{0}) = -\,\partial_{\mathbf{m}}\hat{\psi}(\mathbf{m}, \nabla\mathbf{m}), \tag{2.5a}$$

$$\mathbf{C}^{\mathrm{eq}} = \hat{\mathbf{C}}(\mathbf{m}, \nabla\mathbf{m}; \mathbf{0}, \mathbf{0}) = \partial_{\nabla\mathbf{m}}\hat{\psi}(\mathbf{m}, \nabla\mathbf{m}), \tag{2.5b}$$

while the viscous response has the form

$$\begin{aligned}&\mathbf{k}^{\mathrm{vs}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \dot{\nabla}\mathbf{m})\\ &\quad= \hat{\mathbf{k}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \dot{\nabla}\mathbf{m}) - \hat{\mathbf{k}}(\mathbf{m}, \nabla\mathbf{m}; \mathbf{0}, \mathbf{0}),\end{aligned} \tag{2.6a}$$

$$\begin{aligned}&\mathbf{C}^{\mathrm{vs}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \dot{\nabla}\mathbf{m})\\ &\quad= \hat{\mathbf{C}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \dot{\nabla}\mathbf{m}) - \hat{\mathbf{C}}(\mathbf{m}, \nabla\mathbf{m}; \mathbf{0}, \mathbf{0}),\end{aligned} \tag{2.6b}$$

and satisfies the dissipation inequality to which (2.4) reduces, namely,

$$-\,\mathbf{k}^{\mathrm{vs}}\cdot\dot{\mathbf{m}} + \mathbf{C}^{\mathrm{vs}}\cdot\dot{\nabla}\mathbf{m} \geq 0. \tag{2.7}$$

We now write (2.1) in the form

$$\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m} \times \mathbf{h}^{\mathrm{eff}} + \mathbf{r}, \tag{2.8}$$

where $\mathbf{h}^{\mathrm{eff}}$, the *effective field*, and $\mathbf{r}$, the *relaxation vector*, are defined to be, respectively,

$$\mathbf{h}^{\mathrm{eff}} = \mathbf{b}^{\mathrm{ni}} + \mathbf{k}^{\mathrm{eq}} + \operatorname{div}\mathbf{C}^{\mathrm{eq}} \tag{2.9}$$

[5] Our developments apply also to ferromagnets that are kept immobile, *i.e.*, ferromagnets that are "at mechanical rest" in the terminology of [2]. Broader, but of course less transparent, generalizations of the Gilbert and Landau-Lifshitz equations obtain if one considers *deformable* ferromagnets [1,2] and, as we do in the next two sections, more complex response mappings than in (2.2–2.4). In (2.2), $\dot{\nabla}(\cdot) = \frac{\mathrm{d}}{\mathrm{d}t}(\nabla(\cdot))$; the order of time and space differentiation is immaterial in the case of ferromagnets that are rigid or at mechanical rest (in fact, for those $\dot{\nabla}\mathbf{m} = \nabla\dot{\mathbf{m}}$), but in general it is not so.

and

$$\mathbf{r} = \mathbf{m} \times \mathbf{h}^{\text{vs}}, \quad \mathbf{h}^{\text{vs}} = \mathbf{k}^{\text{vs}} + \operatorname{div} \mathbf{C}^{\text{vs}}, \tag{2.10}$$

with $\mathbf{h}^{\text{vs}}$ the *viscous field*[6]. The generally accepted interpretation of this form of the evolution equation for magnetic torques is that the dynamics of the magnetization vector $\mathbf{m}$ is a precession about the effective field $\mathbf{h}^{\text{eff}}$, a motion eventually damped out to the equilibrium situation that obtains when the relaxation process ends and the term $\mathbf{r}$ vanishes:

$$\mathbf{m} \times \mathbf{h}^{\text{eff}} = \mathbf{0}. \tag{2.11}$$

We postpone commenting this interpretation, and pass to derive from (2.8) the standard Gilbert equation.

The usual, simplest constitutive choices are:
(i) for the free energy,

$$\psi_{\text{S}} = \frac{1}{2}\beta(\mathbf{m} \cdot \mathbf{e})^2 + \frac{1}{2}\alpha\, |\nabla\mathbf{m}|^2, \;\; \alpha > 0, \tag{2.12}$$

with β the *anisotropy-energy modulus* and α the *exchange-energy modulus*, so that the equilibrium response takes the form

$$\mathbf{k}^{\text{eq}} = -\,\beta(\mathbf{m} \cdot \mathbf{e})\mathbf{e}, \tag{2.13a}$$
$$\mathbf{C}^{\text{eq}} = \,\alpha\,\nabla\mathbf{m}; \tag{2.13b}$$

(ii) for the viscous response,

$$\mathbf{k}^{\text{vs}} = -\,\mu\,\dot{\mathbf{m}}, \;\; \mu \geq 0, \tag{2.14a}$$
$$\mathbf{C}^{\text{vs}} = \mathbf{0}, \tag{2.14b}$$

with μ the *interaction-dissipation modulus*; needless to say, (2.14) agree with the dissipation inequality (2.7). With the choices (2.13), relation (2.9) yields

$$\mathbf{h}^{\text{eff}}_{\text{S}} = \mathbf{b}^{\text{ni}} - \,\beta\,(\mathbf{m} \cdot \mathbf{e})\mathbf{e} + \,\alpha\,\Delta\mathbf{m}. \tag{2.15}$$

On the other hand, with (2.14), the relaxation vector defined by (2.10) takes the Gilbert form

$$\mathbf{r}_{\text{G}} = -\,\mu\,\mathbf{m} \times \dot{\mathbf{m}}. \tag{2.16}$$

All in all, when one inserts in (2.8) both the standard effective field $\mathbf{h}^{\text{eff}}_{\text{S}}$ for $\mathbf{h}^{\text{eff}}$ and the Gilbert relaxation $\mathbf{r}_{\text{G}}$ for $\mathbf{r}$, the *standard Gilbert equation* is arrived at:

$$\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m} \times \big(\mathbf{b}^{\text{ni}} - \,\beta\,(\mathbf{m} \cdot \mathbf{e})\mathbf{e} + \,\alpha\,\Delta\mathbf{m}\big) - \mu\,\mathbf{m} \times \dot{\mathbf{m}}. \tag{2.17}$$

The evolution equation (2.8) – when coupled with constitutive equations of type (2.5) and (2.6) consistent with the dissipation inequality (2.7) – constitutes a broad generalization of (2.17). In particular, the standard effective field $\mathbf{h}^{\text{eff}}_{\text{S}}$ in (2.15) and the Gilbert viscous field

$$\mathbf{h}^{\text{vs}}_{\text{G}} = -\,\mu\,\dot{\mathbf{m}} \tag{2.18}$$

[6] Our $\mathbf{h}^{\text{eff}}$ is denoted by $\mathbf{H}$ in [4] and by $\mathbf{H}_{\text{eff}}$ in [6], pp. 182–186.

are generalized to, respectively,

$$\mathbf{h}^{\text{eff}} = \mathbf{b}^{\text{ni}} - \Big(\partial_{\mathbf{m}}\hat{\psi}(\mathbf{m}, \nabla\mathbf{m}) - \,\operatorname{div} \partial_{\nabla\mathbf{m}}\hat{\psi}(\mathbf{m}, \nabla\mathbf{m})\Big) \tag{2.19}$$

and

$$\mathbf{h}^{\text{vs}} = \mathbf{k}^{\text{vs}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \dot{\nabla}\mathbf{m}) + \operatorname{div} \mathbf{C}^{\text{vs}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \dot{\nabla}\mathbf{m}) \tag{2.20}$$

for any thermodynamically compatible assignment of the mappings $\mathbf{k}^{\text{vs}}$ and $\mathbf{C}^{\text{vs}}$, that is, for whatever mappings satisfy

$$\mathbf{k}^{\text{vs}}(\mathbf{m}, \nabla\mathbf{m}; \mathbf{0}, \mathbf{0}) = \mathbf{0}, \tag{2.21a}$$
$$\mathbf{C}^{\text{vs}}(\mathbf{m}, \nabla\mathbf{m}; \mathbf{0}, \mathbf{0}) = \mathbf{0}, \tag{2.21b}$$
$$-\mathbf{k}^{\text{vs}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \dot{\nabla}\mathbf{m}) \cdot \dot{\mathbf{m}} + \mathbf{C}^{\text{vs}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \dot{\nabla}\mathbf{m}) \cdot \dot{\nabla}\mathbf{m} \geq 0. \tag{2.21c}$$

3 Liapounov structures of the Gilbert and Landau-Lifshitz types

In this section we indicate what Liapounov structures are associated with the Gilbert equation in its standard and generalized forms, and with the Landau-Lifshitz equation, a version of the standard Gilbert equation which is also susceptible of generalization, although to a lesser extent. Our first step is to show that the generalized effective field (2.19) can be seen as the variational derivative of a suitably defined effective energy.

3.1 The effective field as a variational derivative

With the use of (2.5), the *equilibrium field*

$$\mathbf{h}^{\text{eq}} = \mathbf{k}^{\text{eq}} + \operatorname{div} \mathbf{C}^{\text{eq}} \tag{3.1}$$

is seen to be the negative of the variational derivative of the free-energy functional:

$$\mathbf{h}^{\text{eq}} = -\delta_{\mathbf{m}}\,\psi\ ^{7}. \tag{3.2}$$

As to the noninertial distance couple $\mathbf{b}^{\text{ni}}$, we note that its form is irrelevant to a discussion of dissipation mechanisms, where only those terms in the evolution equation (2.1) count that are the subject of a constitutive choice restricted by the dissipation principle (2.4): for what it matters to discussing dissipation mechanisms, we could as well put in the sequel $\mathbf{b}^{\text{ni}} = \mathbf{0}$. However, in the general evolution equation (2.1) as well as in its special versions, $\mathbf{b}^{\text{ni}}$ is usually taken to be

$$\mathbf{b}^{\text{ni}} = \mathbf{h}^{\text{ext}} + \mathbf{h}^{\text{mag}}(\mathbf{m}), \tag{3.3}$$

[7] By the variational derivative of a functional of the form $\mathbf{u} \mapsto \int_{\Omega} \phi(\mathbf{u}, \nabla\mathbf{u})$ we mean the associated field operator of Euler-Lagrange, that is,

$$\delta_{\mathbf{u}}\phi = \partial_{u}\phi - \operatorname{div}(\partial_{\nabla\mathbf{u}}\phi).$$

where $\mathbf{h}^{\text{ext}}$ is the *external field* and $\mathbf{h}^{\text{mag}}(\mathbf{m})$, the *magnetostatic field*, is the value at a given magnetization field of the linear Green functional that yields the solution of the Maxwell equations in the quasi-static approximation [1,2][8]. Each of these fields can be seen as a variational derivative: respectively, of the external energy and of the magnetostatic energy, whose densities are

$$\psi^{\text{ext}} = -\,\mathbf{h}^{\text{ext}}\cdot\mathbf{m}, \tag{3.4}$$

$$\psi^{\text{mag}} = -\,\frac{1}{2}\mathbf{h}^{\text{mag}}(\mathbf{m})\cdot\mathbf{m}\,; \tag{3.5}$$

hence,

$$\mathbf{b}^{\text{ni}} = -\delta_{\mathbf{m}}(\psi^{\text{ext}}+\psi^{\text{mag}})\,^{9}. \tag{3.6}$$

Finally, on defining the *effective energy* per unit volume to be

$$\psi^{\text{eff}} = \psi^{\text{ext}}+\psi^{\text{mag}}+\psi\,, \tag{3.7}$$

we have the desired result:

$$\mathbf{h}^{\text{eff}} = -\delta_{\mathbf{m}}\,\psi^{\text{eff}}\,. \tag{3.8}$$

3.2 Liapounov structure of Gilbert type

Take the inner product of the generalized Gilbert equation (2.8) with $\mathbf{m}\times\dot{\mathbf{m}}$, then integrate the resulting relation over the region Ω occupied by the rigid ferromagnet under study. Then, due to relations (3.8) and (2.10), and the fact that, in view of the saturation condition, $\mathbf{m}\times\mathbf{v}\,\cdot\,\mathbf{m}\times\dot{\mathbf{m}} = \mathbf{v}\cdot\dot{\mathbf{m}}$ for each vector $\mathbf{v}$, it turns out that

$$\int_\Omega \mathbf{m}\times\mathbf{h}^{\text{eff}}\cdot\mathbf{m}\times\dot{\mathbf{m}} = -\int_\Omega \delta_{\mathbf{m}}\,\psi^{\text{eff}}\cdot\dot{\mathbf{m}}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\psi^{\text{eff}} + \int_{\partial\Omega}\mathbf{C}^{\text{eq}}\mathbf{n}\cdot\dot{\mathbf{m}}, \tag{3.9a}$$

$$\int_\Omega \mathbf{r}\,\cdot\,\mathbf{m}\times\dot{\mathbf{m}} = \int_\Omega \mathbf{h}^{\text{vs}}\cdot\dot{\mathbf{m}} \tag{3.9b}$$

$$= -\int_\Omega(-\,\mathbf{k}^{\text{vs}}\cdot\dot{\mathbf{m}}+\mathbf{C}^{\text{vs}}\cdot\dot{\nabla}\mathbf{m}) + \int_{\partial\Omega}\mathbf{C}^{\text{vs}}\mathbf{n}\cdot\dot{\mathbf{m}}.$$

Thus, in general, since neither of the boundary integrals on the right sides of relations (3.9) vanishes, neither the power expended by the effective field is totally recoverable nor the power expended by the viscous field is totally dissipated as a consequence of (2.7). However, we note that

$$\int_{\partial\Omega}(\mathbf{C}^{\text{eq}}+\mathbf{C}^{\text{vs}})\mathbf{n}\,\cdot\,\dot{\mathbf{m}} = \int_{\partial\Omega}\mathbf{m}\times\mathbf{C}\mathbf{n}\,\cdot\,\mathbf{m}\times\dot{\mathbf{m}}\,, \tag{3.10}$$

so that, whenever the natural boundary condition

$$\mathbf{m}\times\mathbf{C}\mathbf{n} = \mathbf{0}\quad\text{over}\quad\partial\Omega \tag{3.11}$$

prevails, we arrive at

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\psi^{\text{eff}} + \int_\Omega(-\,\mathbf{k}^{\text{vs}}\cdot\dot{\mathbf{m}}+\mathbf{C}^{\text{vs}}\cdot\dot{\nabla}\mathbf{m})\,. \tag{3.12}$$

Due to the reduced dissipation inequality (2.7), we conclude that, in a smoothly time-dependent magnetization process, the total effective energy

$$\Psi^{\text{eff}} = \int_\Omega\psi^{\text{eff}}$$

plays the role of a Liapounov function, in that

$$\dot{\Psi}^{\text{eff}}\le 0 \tag{3.13}$$

along the system's orbits.

Remark

In the case of the standard Gilbert equation, the effective field $\mathbf{h}_{\text{S}}^{\text{eff}}$ is given by (2.15), whence a simplification in the form of the boundary condition (3.11), which becomes

$$\partial_{\mathbf{n}}\mathbf{m} = \mathbf{0}\quad\text{over}\quad\partial\Omega\,^{10}; \tag{3.14}$$

moreover, since the relaxation vector has the form (2.16), the viscous part (3.9b) of the expended power becomes

$$\int_\Omega \mathbf{r}_{\text{G}}\,\cdot\,\mathbf{m}\times\dot{\mathbf{m}} \,=\, -\,\mu\int_\Omega|\dot{\mathbf{m}}|^2\,. \tag{3.15}$$

Hence, equation (3.12) reduces to

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\psi_{\text{S}}^{\text{eff}} + \mu\int_\Omega|\dot{\mathbf{m}}|^2\,. \tag{3.16}$$

3.3 The Landau-Lifshitz equation and its Liapounov structure

Return to (2.8) with the relaxation vector in the Gilbert form (2.16), namely,

$$\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m}\times\mathbf{h}^{\text{eff}} - \,\mu\,\mathbf{m}\times\dot{\mathbf{m}}. \tag{3.17}$$

It is not difficult to give equation (3.17) the form

$$(1+\gamma^2\mu^2)\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m}\times(\mathbf{h}^{\text{eff}}-\gamma\mu\,\mathbf{m}\times\mathbf{h}^{\text{eff}}); \tag{3.18}$$

the *standard Landau-Lifshitz equation* is the special version of (3.18) which obtains when $\mathbf{h}^{\text{eff}} = \mathbf{h}_{\text{S}}^{\text{eff}}$, with the standard effective field $\mathbf{h}_{\text{S}}^{\text{eff}}$ given by (2.15) combined with (3.3).

The generalized Landau-Lifshitz equation (3.18) can also be written in the form (2.8):

$$\gamma^{-1}\dot{\mathbf{m}} \,=\, \mathbf{m}\times\widetilde{\mathbf{h}}^{\text{eff}}+\mathbf{r}_{\text{L}}\,, \tag{3.19}$$

[8] Our external field is often called the *applied field*, and our magnetostatic field the *demagnetizing field* [3,6].

[9] A proof of differentiability for the functional $\int_\Omega(\psi^{\text{ext}}+\psi^{\text{mag}})$ is found in [12].

[10] Here $\partial_{\mathbf{n}}$ denotes differentiation in the direction of the normal $\mathbf{n}$ to the boundary.

where

$$\begin{aligned}\widetilde{\mathbf{h}}^{\mathrm{eff}} &= \frac{1}{1+\gamma^2\mu^2}\,\mathbf{h}^{\mathrm{eff}},\\ \mathbf{r}_{\mathrm{L}} &= \frac{\gamma\mu}{1+\gamma^2\mu^2}\,\mathbf{h}_\perp^{\mathrm{eff}},\\ \mathbf{h}_\perp^{\mathrm{eff}} &= -\,\mathbf{m}\times(\mathbf{m}\times\mathbf{h}^{\mathrm{eff}})\end{aligned} \tag{3.20}$$

($\mathbf{h}_\perp^{\mathrm{eff}}$, as the definition shows and the notation suggests, is the part of $\mathbf{h}^{\mathrm{eff}}$ orthogonal to $\mathbf{m}$).

In order to derive the Liapounov structure associated with the Landau-Lifshitz equation we note that

$$\begin{aligned}\int_\Omega \mathbf{h}^{\mathrm{eff}}\cdot\dot{\mathbf{m}} &= -\int_\Omega \delta_{\mathbf{m}}\,\psi^{\mathrm{eff}}\cdot\dot{\mathbf{m}}\,,\\ \int_\Omega \mathbf{h}^{\mathrm{eff}}\cdot\mathbf{r}_{\mathrm{L}} &= -\,\frac{\gamma\mu}{1+\gamma^2\mu^2}\int_\Omega |\mathbf{h}_\perp^{\mathrm{eff}}|^2\,.\end{aligned}$$

From these relations, with the use of (3.19, 3.20, 3.9a), and (3.14), we deduce that

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi^{\mathrm{eff}}+\frac{\gamma^2\mu}{1+\gamma^2\mu^2}\int_\Omega |\mathbf{h}_\perp^{\mathrm{eff}}|^2\,; \tag{3.21}$$

and from this the Liapounov relation (3.13) follows (*cf.* [3]), provided that $\mu>0$ (*cf.* (2.14a)) but, other than that, seemingly with no recourse to a dissipation inequality of thermodynamic significance such as (2.7).

Remark

In spite of the claims to the contrary that one repeatedly finds in the literature, the Gilbert and Landau-Lifshitz equations (3.17) and (3.18) are equivalent, both mathematically[11] and physically. In particular, for $(\gamma\mu)\to 0$ (that is, in the limit of vanishing damping) both equations reduce to

$$\gamma^{-1}\dot{\mathbf{m}}\;=\;\mathbf{m}\times\mathbf{h}_{\mathrm{S}}^{\mathrm{eff}}\,; \tag{3.22}$$

for $(\gamma\mu)\to\infty$ (large damping), they both imply that $\dot{\mathbf{m}}\to\mathbf{0}$.

[11] For $\mathbf{1}$ the unit matrix and $\mathbf{M}$ the skew-symmetric matrix uniquely associated with the unit vector $\mathbf{m}$, equations (3.17) and (3.18) can be written as, respectively,

$$(\mathbf{1}+\gamma\mu\,\mathbf{M})^{-1}\gamma^{-1}\dot{\mathbf{m}}\;=\;\mathbf{M}\mathbf{h}^{\mathrm{eff}}$$

and

$$(1+\gamma^2\mu^2)\gamma^{-1}\dot{\mathbf{m}}\;=\;(\mathbf{1}-\gamma\mu\,\mathbf{M})\mathbf{M}\mathbf{h}^{\mathrm{eff}}.$$

On the other hand, for α a real number and $\mathbf{a}$ a unit vector whose associated skew matrix is $\mathbf{A}$, the following inversion formula holds ([2], Appendix A):

$$(\mathbf{1}+\alpha\,\mathbf{A})^{-1}=(1+\alpha^2)^{-1}\left(\mathbf{1}-\alpha\,\mathbf{A}+\alpha^2\,\mathbf{a}\otimes\mathbf{a}\right),$$

where the matrix $\mathbf{a}\otimes\mathbf{a}$ is defined by its action on each vector $\mathbf{v}$: $(\mathbf{a}\otimes\mathbf{a})\mathbf{v}=(\mathbf{a}\cdot\mathbf{v})\mathbf{a}$.

It is instructive to compare the consequences of the Liapounov relations associated with (3.17) and (3.18). Firstly, we deduce from (3.16) and (3.21) that

$$\mu^2\int_\Omega|\dot{\mathbf{m}}|^2=\frac{\gamma^2\mu^2}{1+\gamma^2\mu^2}\int_\Omega|\mathbf{h}_\perp^{\mathrm{eff}}|^2\,. \tag{3.23}$$

In addition, if we agree, as is common practice, that a relaxation process expires when

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\psi^{\mathrm{eff}}\right)\to 0\,, \tag{3.24}$$

then, according to (3.16, 3.24) implies that $\dot{\mathbf{m}}\to\mathbf{0}$, and hence (equivalently in view of (2.16), under saturation conditions) that $\mathbf{r}_{\mathrm{G}}$ tends to null; while, with (3.21, 3.24) implies that $\mathbf{h}_\perp^{\mathrm{eff}}\to\mathbf{0}$, and hence that $\mathbf{r}_{\mathrm{L}}$ tends to null (*cf.* $(3.20)_2$). In fact, under saturation conditions, it is easy to show that, if the Landau-Lifshitz equation holds, then $(\mathbf{r}_{\mathrm{L}}=\mathbf{0})\Rightarrow(\mathbf{r}_{\mathrm{G}}=\mathbf{0})$ (provided $\mu\neq 0$); and that, if the standard Gilbert equation holds, the converse implication holds true.

4 Dissipation mechanisms compared

The Remark that ends the previous section exposes the main conceptual difference between the Gilbert and Landau-Lifshitz formats: the first links relaxation to dissipation, the second to the expected equilibria; and our discussion in Section 2 indicates that dissipation mechanisms and equilibrium states are both determined by constitutive anticipations being, in general, mutually independent.

If the dissipation is chosen to have the standard Gilbert form (2.16), then the Gilbert and Landau-Lifshitz approaches to relaxation are equivalent. But a Gilbert-type format should be preferred when looking for general descriptions of the evolution of a ferromagnet, because it conveniently keeps dissipation and equilibrium separate: within a Gilbert approach, the constitutive prescriptions can be modified one at a time, and the consequences of each modification clearly identified.

This issue is of special importance in the light of known examples of nonuniqueness in dynamics when certain singular solutions are found possible in statics (see the Remark below): it is natural to ask whether there are thermodynamically admissible constitutive modifications such as to select among the possible evolution processes those being physically plausible. In the remain of this section we discuss certain modifications of this nature that have been recently proposed, leading to the consideration of dissipative terms other than Gilbert's [2] or/and of terms that change the set of expected equilibrium states [3,4].

The first proposal, which we develop within the generalized Gilbert format of Section 2, is intended to introduce exchange dissipation into the standard model. The other two proposals were both advanced within the Landau-Lifschitz format. The model equations that are arrived at are worth-studying, even if the derivations are questionable for the reasons given in the Introduction and here above. We briefly indicate how these equations are deducible within a Gilbert format that needs, however, some further generalization with respect to that of Section 2.

Remark

The model equation

$$\gamma^{-1}\dot{\mathbf{m}} = \alpha\,\mathbf{m}\times\,\Delta\mathbf{m} - \,\mu\,\mathbf{m}\times\dot{\mathbf{m}}\,, \tag{4.1}$$

a stripped version of the standard Gilbert equation (2.17), can be given the equivalent form analyzed in [2], namely,

$$\mu\dot{\mathbf{m}} - \gamma^{-1}\mathbf{m}\times\dot{\mathbf{m}} = \alpha(\Delta\mathbf{m} + |\nabla\mathbf{m}|^2\mathbf{m}),\ \ |\mathbf{m}| = 1; \tag{4.2}$$

this form makes evident the relation of (4.1) to the *harmonic-map equation*

$$\Delta\mathbf{m} + |\nabla\mathbf{m}|^2\mathbf{m} = \mathbf{0}\,, \quad |\mathbf{m}| = 1\,, \tag{4.3}$$

a well-known, physically oversimplified model for the statics of nematic *liquid crystals*. Equation (4.1) shares with equation (4.3) certain stationary solutions with finite energy (in dimension 3) and a *point singularity*: say,

$$\mathbf{m}(x) = |x - x_o|^{-1}(x - x_o)\,;$$

if a singular stationary solution of this type is taken as the initial datum, then another solution of (4.1) may happen to evolve in time [13]. Moreover, an example of Chang, Ding, and Ye [14] for the equation

$$\mu\,\dot{\mathbf{m}} = \alpha(\Delta\mathbf{m} + |\nabla\mathbf{m}|^2\mathbf{m}) \tag{4.4}$$

on a right circular cylinder suggests that a *line singularity* with finite energy might develop in finite time also for equation (4.1) (or, perhaps, for Eq. (2.17) itself); the question also arises as to how such a solution would continue after a finite-energy singularity is formed (see the discussions and the literature cited in [2] and [15]).

4.1 Exchange dissipation

The terms in (2.1) relevant to a discussion of dissipation mechanisms are $\mathbf{m}\times\mathbf{k}$, the *interaction couple*, and $\mathbf{m}\times\operatorname{div}\mathbf{C}$, the *exchange couple*. We now indicate briefly their physical significance.

When a ferromagnet is regarded as the composition of two interacting continua (recall the second paragraph of Sect. 2), the interaction couple $\mathbf{m}\times\mathbf{k}$ must enter into the balance of torques peculiar of the magnetic constituent, equation (2.1); its equilibrium part $\mathbf{m}\times\mathbf{k}^{\rm eq}$ accounts primarily for the *anisotropy energy* (compare (2.5a) with its standard special case (2.13a))[12]; its viscous part $\mathbf{m}\times\mathbf{k}^{\rm vs}$ accomodates for the *relativistic interaction* between the magnetic moments in the crystal, and has been given the form (2.16) by Gilbert and the form (3.20b) by Landau and Lifshitz ([5], p. 109).

The reason why we call $\mathbf{m}\times\operatorname{div}\mathbf{C}$ the exchange couple becomes evident when we compare the prescription (2.5b) for the equilibrium part $\mathbf{C}^{\rm eq}$ of the couple stress $\mathbf{C}$ with its standard special case (2.13b): $\mathbf{C}$ accounts for the contact action peculiar of the magnetic constituent, whose distance-action equivalent is $\operatorname{div}\mathbf{C}$; the constitutive relation (2.13b) connects $\mathbf{C}^{\rm eq}$ with the magnetization gradient $\nabla\mathbf{m}$ and the related standard form $\frac{1}{2}\alpha\,|\nabla\mathbf{m}|^2$ of the *exchange energy*[13]. It is therefore natural to associate with the viscous part $\mathbf{m}\times\operatorname{div}\mathbf{C}^{\rm vs}$ a dissipation mechanism, additional to the relativistic interaction envisaged by Gilbert and attributed to exchange effects.

The simplest constitutive prescription compatible with the reduced dissipation inequality (2.7) is

$$\mathbf{C}^{\rm vs} = \tau\,\dot{\nabla}\mathbf{m}\,,\ \ \tau > 0\,. \tag{4.5}$$

When the viscous response is so augmented, the relaxation power (3.9b) becomes

$$\int_\Omega \mathbf{r}\,\cdot\,\mathbf{m}\times\dot{\mathbf{m}} = -\int_\Omega(\mu\,|\dot{\mathbf{m}}|^2 + \tau\,|\dot{\nabla}\mathbf{m}|^2) + \int_{\partial\Omega}\partial_{\mathbf{n}}(\frac{1}{2}\,\tau\,|\dot{\mathbf{m}}|^2) \tag{4.6}$$

(*cf.* (3.15)), while relation (3.16) takes the form

$$0 = \frac{\rm d}{{\rm d}t}\int_\Omega \psi_{\rm S}^{\rm eff} + \int_\Omega(\,\mu\,|\dot{\mathbf{m}}|^2 + \tau\,|\dot{\nabla}\mathbf{m}|^2)\,. \tag{4.7}$$

Remark

The additional dissipation (4.5) translates into the following additional term on the right side of the Gilbert equation: $+\,\tau\,\mathbf{m}\times\Delta\dot{\mathbf{m}}$; in particular, the model equation (4.1) becomes

$$\gamma^{-1}\dot{\mathbf{m}} = \alpha\,\mathbf{m}\times\,\Delta\mathbf{m} - \,\mu\,\mathbf{m}\times\dot{\mathbf{m}} + \,\tau\,\mathbf{m}\times\Delta\dot{\mathbf{m}}\,. \tag{4.8}$$

To our knowledge, an existence proof for equation (4.8) has not been obtained so far. However, the regularizing effect of exchange dissipation is expedient to prove existence of global-weak solutions to (4.2) in the manner of [2]: the main idea of the proof is to modify equation (4.2) by introducing two positive small parameters, ϵ and τ:

$$\mu\,\dot{\mathbf{m}} - \gamma^{-1}\mathbf{m}\times\dot{\mathbf{m}} - \tau\,\Delta\dot{\mathbf{m}} = \alpha(\Delta\mathbf{m} - \epsilon^{-1}(|\mathbf{m}|^2 - 1)\mathbf{m}); \tag{4.9}$$

[12] Anisotropy energy "is ... mainly related to interactions of electron orbitals with the potential created by the hosting lattice" ([6], p. 129); that is to say, anisotropy energy accounts for the interaction of the magnetic constituent with the mechanical constituent (the hosting lattice).

[13] Another quotation from [6], p. 129, is appropriate to substantiate our interpretation of the exchange energy as peculiar of the magnetic constituent: the hosting lattice is now given no role, "exchange derives from the combination of the electrostatic coupling between electron orbitals and the necessity to satisfy the Pauli exclusion principle. It results in spin-spin interactions that favor *long-range spin ordering over macroscopic distances*" (italics are ours).

the $\tau-$regularization allows to solve (4.9) as an ODE in an appropriate function space, while the $\epsilon^{-1}-$penalization replaces the saturation constraint; the existence result for equation (4.4) is obtained by passing to the limit for $\epsilon,\, \tau\, \to 0$.

4.2 Pseudo exchange dissipation

In [4] Baryakthar *et al.* lay down the Landau-Lifshitz equation (3.19), and interpret it just as we did for the general equation (2.8) (*cf.* the sentence after (2.10)); in their view, as is typical within the Landau-Lifshitz format, the expression chosen for $\mathbf{r}_{\rm L}$ embodies a specific *constitutive* visualization of the expected equilibria[14].

As anticipated in the Introduction, the authors of [4] take the Landau-Lifshitz form $(3.20)_2$ of the relaxation vector responsible for various inconsistencies, that they list and discuss, of the predictions based on the phenomenological theory of relaxation processes in (ferro)magnets with the corresponding predictions based on quantum mechanics, as well as with the results of certain experiments. They surmise that the inconsistencies are inevitable in standard models like Landau-Lifshitz' or Gilbert's, where dissipation processes of exchange origin are not properly accounted for; and they propose a variant of the Landau-Lifschitz relaxation vector, a variant they claim to yield more accurate predictions. However, since they use a Landau-Lifshitz format, the only dissipation mechanism they actually include is Gilbert's; their proposal basically amounts to change the characterization of the equilibrium set.

To be specific, consider the standard Landau-Lifshitz equation. According to [4], the effective field $\mathbf{h}_{\rm S}^{\rm eff}$ should be replaced by, say,

$$(\mathbf{h}_{\rm S}^{\rm eff} - \lambda\, \Delta\mathbf{h}_{\rm S}^{\rm eff}), \quad \lambda > 0\,.$$

Consequently, the equilibrium magnetization fields solves

$$\mathbf{m} \times \left(\mathbf{h}_{\rm S}^{\rm eff} - \lambda\, \Delta\mathbf{h}_{\rm S}^{\rm eff}\right) = \mathbf{0}\,, \tag{4.10}$$

rather than

$$\mathbf{m} \times \mathbf{h}_{\rm S}^{\rm eff} = \mathbf{0}\,. \tag{4.11}$$

Remark

When the effective field is modified as above, the model equation (4.1) becomes

$$\gamma^{-1}\dot{\mathbf{m}} = \alpha\, \mathbf{m} \times (\Delta\mathbf{m} - \lambda\, \Delta\Delta\mathbf{m}) - \,\mu\, \mathbf{m} \times \dot{\mathbf{m}}\,. \tag{4.12}$$

The powerful regularizing effect of the biharmonic term allows to show that there are indeed global weak solutions to (4.12), and that, moreover, they solve (4.1) in the limit $\lambda \to 0$ [16].

[14] To quote from [4], p. 619, $\mathbf{r}_{\rm L}$ “describes a magnetization distribution approaching its equilibrium state”.

4.3 Dry-friction dissipation

The proposal in [3] is to include a dissipation term additional to Gilbert's in order to account for the “slip-stick” motion of domain walls[15]. For the model equation (4.1), the inclusion of the simplest additional term of this type leads to

$$\gamma^{-1}\dot{\mathbf{m}} = \alpha\, \mathbf{m} \times\, \Delta\mathbf{m} -\, \mu\, \mathbf{m} \times \dot{\mathbf{m}} + \eta\, \mathbf{m} \times \mathbf{f}(\dot{\mathbf{m}})\,, \quad \eta > 0\,, \tag{4.13}$$

where $\mathbf{f}(\mathbf{v})$, the *dry-friction mapping*, is chosen to be the subgradient of the potential $|\mathbf{v}|$:

$$\mathbf{f}(\mathbf{v}) = -\,\partial_{\mathbf{v}}|\mathbf{v}|\,, \tag{4.14}$$

and hence

$$-\mathbf{f}(\dot{\mathbf{m}}) = \,|\dot{\mathbf{m}}|^{-1}\dot{\mathbf{m}} \quad \text{if}\;\; \dot{\mathbf{m}} \neq \mathbf{0}\,, \;\; -\mathbf{f}(\mathbf{0}) \in \{\, \mathbf{v} \,|\, |\mathbf{v}| \leq 1\} \cdot \tag{4.15}$$

Accordingly, the relaxation vector (2.10) takes the form

$$\mathbf{r} = \mathbf{m} \times (-\mu\, \dot{\mathbf{m}} + \eta\, \mathbf{f}(\dot{\mathbf{m}}))\,, \;\; \dot{\mathbf{m}} \neq \mathbf{0}\,, \tag{4.16}$$

and the associated relaxation power (3.9b) is

$$\int_\Omega \mathbf{r}\, \cdot\, \mathbf{m} \times \dot{\mathbf{m}} = -\int_\Omega (\,\mu\, |\dot{\mathbf{m}}|^2 + \eta\, |\dot{\mathbf{m}}|)\,. \tag{4.17}$$

However, this changes to the standard Gilbert equation and its consequences are not the only outcome of adding a dry-friction term to (4.1), because the equilibrium set is modified as well: in fact, equilibrium is now guaranteed whenever

$$\mathbf{0} = \,\mathbf{m} \times (\alpha\, \Delta\mathbf{m} + \eta\, \mathbf{f}(\mathbf{0})) \tag{4.18}$$

holds for $\mathbf{m}$ unimodular and $\mathbf{f}(\mathbf{0})$ compliant with the second of (4.15).

Remark

With a view toward making greater the insight offered by the above example of dry-friction dissipation, more general constitutive equations than (4.14) must be sought, consistent with rate-independence. Just as is done for similar reasons in classical theories of plasticity, one may think of constitutive mappings $\mathbf{k}(\mathbf{v})$ being *homogeneous of degree zero* and continuous over the set of all nonnull vectors, with at most a jump discontinuity at $\mathbf{v} = \mathbf{0}$. Given such a mapping, it is easy to show that there is another mapping $\widetilde{\mathbf{k}}$, defined over the collection of all unit vectors and such that $\mathbf{k}(\mathbf{v}) = \widetilde{\mathbf{k}}(|\mathbf{v}|^{-1}\mathbf{v})$. However, it would not be possible to replace the constitutive equation (2.2a) by, say,

$$\hat{\mathbf{k}}(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \dot{\nabla}\mathbf{m}) = \hat{\mathbf{k}}_1(\mathbf{m}, \nabla\mathbf{m}; \dot{\mathbf{m}}, \dot{\nabla}\mathbf{m}) + \hat{\mathbf{k}}_2(|\dot{\mathbf{m}}|^{-1}\dot{\mathbf{m}}) \tag{4.19}$$

[15] It is suggested in [3] that domain walls are pinned by a number of magnetic defects (impurities, dislocations, and others), and that reported to occur due to removal-restauration of wall pinning due to magnetic defects.

and yet to expect to obtain *gratis* representations of the type (2.5–2.7) from the requirement that the constitutive equations (4.19, 2.2b, 2.3), be consistent with the dissipation principle (2.4): in fact, the standard argument leading to those representations demands continuity at $(\mathbf{0}, \mathbf{0})$ of the mapping $\hat{\mathbf{k}}(\mathbf{m}, \nabla\mathbf{m}; \cdot, \cdot)$ (see Appendix B of [2]).

5 Conclusions

The standard Gilbert equation for the evolution of the magnetization vector in a saturated ferromagnet includes one dissipation term, called relativistic due to its proposed microscopic explanation, and one exchange-energy term.

Here it has been shown how a generalized Gilbert equation can be derived within the framework of a dynamic version of micromagnetics. This generalized equation can include, in addition to relativistic dissipation, various other dissipation mechanisms, all compatible with the principles of continuum thermodynamics; it can also include terms accounting for higher-order gradient energies (the standard exchange energy is a quadratic form in the first spatial gradient of magnetization).

It has also been shown that the modeling of evolving magnetization patterns in a ferromagnet calls for a delicate interplay between physics and mathematical analysis. Generally speaking, additional energies change the collection of equilibrium solutions to a given evolution equation, while additional dissipation mechanisms affect relaxation toward equilibrium; in principle, such additions can be expedient to select the smoothest and/or physically most significant solutions among the many (perhaps, infinitely many) weak solutions of the original equation. The simplest prescriptions of second-order gradient energy, exchange dissipation and dry-friction dissipation have been given special attention; in particular, their mathematical consequences have been indicated for an especially simple model equation. A research topic currently under study along these lines is to find out whether these (or other) energetic and dissipative terms would rule out the development with time of finite-energy, low-dimensional singularities; and, in cases when they do not rule them out, how a solution would continue after singularities are formed.

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References

1. A. DeSimone, P. Podio-Guidugli, Arch. Rational Mech. Anal. **136**, 201 (1996).
2. M. Bertsch, P. Podio-Guidugli, V. Valente, Ann. Mat. Pura Appl. **CLXXVII** (2000).
3. A. Visintin, Physica B **233**. 365 (1997).
4. V.G. Baryakthar, B.A. Ivanov, A.L. Sukstanskii, E. Yu. Melikhov, Phys. Rev. B **56**, 619 (1997).
5. L. Landau, E. Lifshitz, Phys. Z. Sowjet. **8**, 153 (1935). Reprinted on pp. 101-114 of *Collected Papers of L.D. Landau*, edited by D. ter Haar (Pergamon Press, 1965).
6. G. Bertotti, *Hysteresis in Magnetism* (Academic Press, 1998).
7. A. DeSimone, P. Podio-Guidugli, Meccanica **30**, 629 (1995).
8. A. DeSimone, P. Podio-Guidugli, Math. Models Methods Appl. Sci. **7**, 477 (1997).
9. T.L. Gilbert, Phys. Rev. **100**, 1243 (1955).
10. W.F. Brown, *Magnetoelastic Interactions* (Springer-Verlag, 1966).
11. W.F. Brown, *Micromagnetics* (Krieger, 1978).
12. G. Anzellotti, S. Baldo, A. Visintin, Appl. Math. Optim. **23**, 171 (1991).
13. F. Alouges, A. Soyeur, Nonlinear Anal. Theory, Meth. Appl. **18**, 1071 (1992).
14. K.-C. Chang, W.-Y. Ding, R. Ye, J. Differential Geometry **36**, 507 (1992).
15. M. Bertsch, R. Dal Passo, R. van der Hout (unpublished).
16. P. Podio-Guidugli, V. Valente, Nonlinear Anal. Theory, Meth. Appl. (to be published).